ON THE K-THEORY OF GROUPS WITH FINITE ASYMPTOTIC DIMENSION

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ABSTRACT. It is proved that the assembly maps in algebraic K- and L-theory with respect to the family of finite subgroups is injective for groups Γ with finite asymptotic dimension that admit a finite model for $\underline{E}\Gamma$. The result also applies to certain groups that admit only a finite dimensional model for $\underline{E}\Gamma$. In particular, it applies to discrete subgroups of virtually connected Lie groups.

INTRODUCTION

Assembly maps in algebraic K- and L-theory are designed to study the K- and L-theory of group rings $R[\Gamma]$, which contain important geometric information about manifolds with fundamental group Γ. Similarly, the Baum-Connes map is used to analyze the topological K-theory of the reduced C^*-algebra of Γ. An important class of groups that has been studied in recent years, by topologists and analysts alike, is the class of discrete groups with *finite asymptotic dimension*. Yu proved the Novikov conjecture for groups with finite asymptotic dimension that admit a finite classifying space [Yu98]. He achieved this by using controlled techniques to prove the injectivity of the Baum-Connes map for such groups. Later on, Higson was able to remove the finite classifying space assumption [Hig00]. In [Bar03b], the first author proved the algebraic K- and L-theory versions of Yu's work by developing a squeezing theorem for higher algebraic K-theory to use with the approach established in [Yu98]. For algebraic K-theory, this was also achieved in [CG04a].

The purpose of this paper is to extend the results from [Bar03b] by relaxing the finite $B\Gamma$ assumption to allow for groups Γ with torsion. Specifically, we prove the following theorem.

Theorem A. *Let Γ be a discrete group and R a ring. Assume that there is a finite dimensional Γ-CW-model for the universal space for proper Γ-actions, $\underline{E}\Gamma$, and assume that there is a Γ-invariant metric on $\underline{E}\Gamma$ such that $\underline{E}\Gamma$ is uniformly $\mathcal{F}in$-contractible, is a complete proper path metric space and has finite asymptotic dimension. Then the assembly map,*

$$(0.1) \qquad H_*^{\Gamma}(\underline{E}\Gamma; \mathbf{K}_R) \to K_*(R[\Gamma]),$$

in algebraic K-theory, is a split injection.

The notion of uniform $\mathcal{F}in$-contractibility is a strengthening of uniform contractibility and is introduced in Definition 1.1. The asymptotic dimension of a metric space was introduced by Gromov [Gro93] and is reviewed in Section 2.

Note that Theorem A has interesting consequences for Whitehead groups. The classical assembly map

$$(0.2) \qquad\qquad H_n(B\Gamma; \mathbf{K}_R) \to K_n(R[\Gamma])$$

considered in [Bar03b, CG04a], factors through the assembly map (0.1) via

$$(0.3) \qquad\qquad H_n(B\Gamma; \mathbf{K}_R) \cong H_n^\Gamma(E\Gamma; \mathbf{K}_R) \to H_n^\Gamma(\underline{E}\Gamma; \mathbf{K}_R).$$

In particular, the cokernel of (0.2) contains the cokernel of (0.3) in the situation of Theorem A. This cokernel can be evaluated using an Atiyah-Hirzebruch spectral sequence (see [DL98, Theorem 4.7]), or computed rationally using the equivariant Chern character from [Lüc02, Theorem 0.3], [LR05, Theorem 173]. If $R = \mathbb{Z}$ and $n = 1$, then the cokernel of (0.2) is the Whitehead group. In this way, Theorem A implies non-vanishing results for the Whitehead group.

In Section 3 it is shown that the assumptions of Theorem A are satisfied for all discrete subgroups of Lie groups with a finite number of components. Also, if there is a cocompact Γ-CW-model for $\underline{E}\Gamma$, then, with any Γ-invariant metric, it will be quasi-isometric to Γ when Γ is equipped with a word length metric. By Lemma 1.5, such a model is uniformly $\mathcal{F}in$-contractible. Thus, Theorem A implies the following corollary.

Corollary. *Let Γ be a group and assume that one of the following two conditions is satisfied:*

   (i) Γ is a discrete subgroup of a virtually connected Lie group.
   (ii) Γ has finite asymptotic dimension and admits a cocompact Γ-CW-model for $\underline{E}\Gamma$.

Then the assembly map (0.1) in algebraic K-theory is split injective for every ring R.

The techniques used to prove Theorem A allow for a very similar proof of the corresponding result in L-theory if *ultimate lower quadratic L-theory, $L_*^{\langle -\infty \rangle}$,* is used. The only difference is that the compatibility of L-theory with infinite products is only known under an additional K-theory assumption. This forces the extra hypothesis in the following theorem.

Theorem B. *Let Γ be a discrete group and R a ring with involution. Assume that there is a finite dimensional Γ-CW-model for the universal space for proper Γ-actions, $\underline{E}\Gamma$, and assume that there is a Γ-invariant metric on $\underline{E}\Gamma$ such that $\underline{E}\Gamma$ is uniformly $\mathcal{F}in$-contractible, is a complete proper path metric space and has finite asymptotic dimension. Further assume that for every finite subgroup G of Γ, the group $K_{-i}(R[G])$ vanishes for sufficiently large i. Then the assembly map,*

$$(0.4) \qquad\qquad H_*^\Gamma(\underline{E}\Gamma; \mathbf{L}_R) \to L_*^{\langle -\infty \rangle}(R[\Gamma]),$$

in L-theory, is a split injection.

As in the K-theory case, Theorem B can be used to obtain non-vanishing results for the cokernel of the assembly map

$$H_*(B\Gamma; \mathbf{L}_R) \to L_*^{\langle -\infty \rangle}(R[\Gamma]).$$

It is well-known that the Novikov conjecture on the homotopy invariance of higher signatures is implied by the rational injectivity of this map. Clearly, Theorem B also implies the following result.

Corollary. *Let Γ be a group and assume that one of the following two conditions is satisfied:*

> *(i) Γ is a discrete subgroup of a virtually connected Lie group.*
> *(ii) Γ has finite asymptotic dimension and admits a cocompact Γ-CW-model for $\underline{E}\Gamma$.*

Then the assembly map (0.4) in L-theory is split injective for every ring R with involution such that for every finite subgroup G of Γ, the group $K_{-i}(R[G])$ vanishes for sufficiently large i.

It is interesting to compare the finiteness conditions in Theorems A and B with the finiteness assumption in the rational injectivity result of Bökstedt-Hsiang-Madsen [BHM93] for the assembly map (0.2) (with $R = \mathbb{Z}$), where it is assumed that the integral homology, $H_n(B\Gamma; \mathbb{Z})$, is finitely generated for every n. The only other injectivity results that we are aware of that apply to all subgroups of virtually connected Lie groups are those for the Baum-Connes assembly map by Kasparov [Kas88] and Higson [Hig00] (compare [HR00, Section 4]). This also implies the rational injectivity of (0.4) for $R = \mathbb{Z}$. It should also be noted that Ferry-Weinberger [FW91] prove the Novikov conjecture and the rational injectivity of (0.2) (with $R = \mathbb{Z}$) for fundamental groups of non-positively curved manifolds that are not necessarily compact. By [Bar03a], our results are in accordance with the Farrell-Jones conjecture [FJ93]. For more information about the Baum-Connes and Farrell-Jones conjectures we recommend [LR05].

In Sections 8 and 9, Theorems 8.1 and 9.1 are proven which are slightly stronger than the theorems stated in this introduction. There we will be considering assembly maps for an additive category with a Γ-action, as introduced in [BR05]. For example, this more general setup enables one to study the K- and L-theory of crossed product rings (see [BR05, Section 6]). Theorems A and B follow from Theorems 8.1 and 9.1 respectively, by using the category of finitely generated free R-modules with the trivial action of Γ.

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1. UNIFORM CONTRACTIBILITY

Let A be a subset of a metric space X, and let $R > 0$. Denote by A^R the set of all points x in X for which $d(x, A) \leq R$. If $A = \{x\}$ consists of just one point, then we will abbreviate $x^R = \{x\}^R$.

Definition 1.1 (Uniformly $\mathcal{F}in$-contractible). Let Γ act by isometries on a metric space X. Then X is said to be *uniformly $\mathcal{F}in$-contractible* if for every finite subgroup G of Γ and every $R > 0$ there is an $S > 0$ such that the following holds: If B is a G-invariant subset of X of diameter less than R, then the inclusion, $B \to B^S$, is G-equivariantly null homotopic. In particular, B^S contains a fixed point for the action of G.

Remark 1.2 (Uniformly contractible). If Γ is the trivial group, or more generally if Γ is torsion-free, then X is uniformly $\mathcal{F}in$-contractible if and only if X is *uniformly contractible*. That is, for every $R > 0$ there is an $S > R$ such that for every $x \in X$, the inclusion, $x^R \to x^S$, is null homotopic.

Let X be a metric space with an isometric action of a finite group G. Let $q\colon X \to X/G$ denote the quotient map. Then

$$d_{X/G}(y, y') = d_X(q^{-1}(y), q^{-1}(y')), \quad y, y' \in X/G$$

defines a metric on X/G. We will always consider quotients of metric spaces by an isometric action of a finite group as metric spaces using this metric.

Notation 1.3. Let X be a space with an action of a group G, and let $\mathcal{S}$ be a collection of subgroups of G. Define $X^{\mathcal{S}}$ to be the union of all fixed sets X^H, where H is in $\mathcal{S}$. If $\mathcal{S} = \{G\}$, then $X^{\mathcal{S}}$ is simply X^G. If $\mathcal{S}$ is closed under conjugation by elements of G, then $X^{\mathcal{S}}$ is a G-invariant subspace of X.

Lemma 1.4. *Let X be a metric space that is uniformly $\mathcal{F}in$-contractible with respect to an isometric action of a group Γ. If G is a finite subgroup of Γ and $\mathcal{S}$ is a collection of subgroups of G that is closed under conjugation by G, then the quotient $X^{\mathcal{S}}/G$ is uniformly contractible.*

Proof. Let $R > 0$ be given. Since X is assumed to be uniformly $\mathcal{F}in$-contractible and G is finite, there is an $S > 0$ such that for every subgroup H of G and every H-invariant subset $B \subset X$ of diameter less than or equal to $2R|G|$, the inclusion, $B \to B^S$, is H-equivariantly null homotopic.

Let $y \in X^{\mathcal{S}}/G$ and $x \in q^{-1}(y)$, where $q\colon X^{\mathcal{S}} \to X^{\mathcal{S}}/G$ is the quotient map. Let H be the subgroup of G consisting of all $g \in G$ for which there are $g_1, \ldots, g_n \in G$ such that $g_1 = e$, $g_n = g$ and $d(g_i x, g_{i+1} x) \le 2R$, for $i = 1, \ldots, n - 1$. Then the diameter of $B = Hx^R \subset X$ is bounded by $2R|G|$. Therefore, the inclusion $B \to B^S$ is H-equivariantly null homotopic. In particular, there is a point $z \in B^S$ that is fixed by H. For $g \in G - H$, $gB \cap B = \emptyset$. Therefore, the inclusion, $G \cdot B \to G \cdot B^S$, is G-equivariantly homotopic to a map that sends gB to gz. By G-equivariance, this homotopy can be restricted to $X^{\mathcal{S}}$, which induces the required null homotopy on the quotient. $\qquad\square$

Lemma 1.5. *Let Γ be a group such that there is a finite Γ-CW-model, X, for $\underline{E}\Gamma$. Let d be a Γ-invariant metric on X. Then X is uniformly $\mathcal{F}in$-contractible.*

Proof. Let X be a finite Γ-CW-model for $\underline{E}\Gamma$. That is, there is a proper Γ-CW-complex X with finitely many Γ-cells such that X^G is contractible for every finite subgroup G of Γ and empty otherwise. Note that X is a locally compact space since the Γ-action is cocompact and proper, and for every finite subgroup G of Γ, X is G-equivariantly contractible.

Let $R > 0$ be given. If B is a finite G-invariant subcomplex of X of diameter less than R, then there is an $S = S(B, G)$ such that $B \to B^S$ is G-equivariantly null homotopic. If $\gamma \in \Gamma$, then $\gamma B \to \gamma B^S$ is $G^\gamma = \gamma G \gamma^{-1}$-equivariantly null homotopic since the metric is Γ-invariant. Consider the set of all pairs (B, G), where G is a finite subgroup of Γ and B is a G-invariant subcomplex of X whose diameter is less than R. On this set, $\gamma \in \Gamma$ acts by sending $(B, G) \mapsto (\gamma B, G^\gamma)$. Since the Γ-action on X is proper, the quotient by this action is finite. Therefore, we can choose S independent of B. (In fact, S can be chosen independent of both B and G.) $\qquad\square$

2. Asymptotic dimension

Let $\mathcal{U}$ be an open cover of the metric space X. The cover $\mathcal{U}$ is called *locally finite* if every compact subset of X meets only finitely many members of $\mathcal{U}$. The

dimension of an open cover $\mathcal{U}$ is defined to be the smallest number n such that each x in X is contained in at most $n+1$ members of $\mathcal{U}$. This is also the dimension of the associated simplicial complex $|\mathcal{U}|$. If the diameters of the open sets in $\mathcal{U}$ are uniformly bounded, then $\mathcal{U}$ will be called a *bounded cover*. The *asymptotic dimension* [Gro93, p.28] of X is the smallest integer n such that for any $R > 0$, there exists an n-dimensional bounded cover $\mathcal{U}$ of X whose Lebesgue number is at least R.

Lemma 2.1. *Let X be a proper metric space of finite asymptotic dimension n. Then for every α there exists a locally finite n-dimensional bounded cover $\mathcal{U}$ of X whose Lebesgue number is at least α.*

Proof. Let $\mathcal{U}$ be an n-dimensional bounded cover of X such that for every $x \in X$ there is $U \in \mathcal{U}$ that contains the closed ball x^R. For each $U \in \mathcal{U}$, let $U^{-\alpha}$ be the open set consisting of all points $x \in X$ for which $x^\alpha \subset U$. Then $\mathcal{U}^{-\alpha} = \{U^{-\alpha} \mid U \in \mathcal{U}\}$ is an open cover of X. Since X is a proper metric space, we can find a subcollection $\mathcal{U}_0 \subseteq \mathcal{U}$ and an open set $U' \subset U^{-\alpha}$ for each $U \in \mathcal{U}_0$ such that $\{U' \mid U \in \mathcal{U}_0\}$ is a locally finite set. For each $U \in \mathcal{U}_0$, let U'' be the interior of $(U')^\alpha$. The collection $\mathcal{U}'' = \{U'' \mid U \in \mathcal{U}\}$ is also locally finite. Since $U'' \subset U$ for $U \in \mathcal{U}_0$, $\mathcal{U}''$ is a bounded cover of X of dimension at most n. By construction, the Lebesgue number of $\mathcal{U}''$ is at least α. $\square$

Recall the definition of $X^{\mathcal{S}}$ in Notation 1.3.

Lemma 2.2. *Let X be a metric space with a proper isometric action of the group Γ. Let G be a finite subgroup of Γ. Let $\mathcal{S}$ be a collection of subgroups of G that is closed under conjugation by G. If X has finite asymptotic dimension, then the quotient $X^{\mathcal{S}}/G$ has finite asymptotic dimension.*

Proof. Let n be the asymptotic dimension of X, and let $R > 0$ be given. Because the asymptotic dimension of a subspace is bounded by the asymptotic dimension of the ambient space, there exists an n-dimensional bounded cover $\mathcal{U}$ of $X^{\mathcal{S}}$ whose Lebesgue number is at least R. Let $p \colon X^{\mathcal{S}} \to X^{\mathcal{S}}/G$ be the quotient map. Define $p_*\mathcal{U} = \{p(U) \mid U \in \mathcal{U}\}$. It is easy to check that $p_*\mathcal{U}$ is a bounded cover whose Lebesgue number is at least R. For every $x \in X^{\mathcal{S}}/G$, $p^{-1}(x)$ contains no more than $|G|$ points. Therefore, the dimension of $p_*\mathcal{U}$ is at most $(n+1)|G| - 1$. $\square$

3. Discrete subgroups of Lie groups

In this section it is proven that discrete subgroups of virtually connected Lie groups satisfy the assumptions of Theorems 8.1 and 9.1. Let Γ be a discrete subgroup of a virtually connected Lie group G. If K is a maximal compact subgroup of G, then G/K is a finite dimensional Γ-CW complex that is a model for the universal proper Γ-space $\underline{E}\Gamma$ [Lüc05, Theorem 4.4]. Furthermore, there exists a G-invariant Riemannian metric on G/K (compare Lemma 3.8 (ii) below). By [CG04b, Section 3] and [Ji04, Proposition 3.3], G/K has finite asymptotic dimension. [1] Therefore, we must prove the following result.

Proposition 3.1. *Let Γ be a discrete subgroup of a Lie group G with finitely many components. Let K be a maximal compact subgroup of G. Then the Γ-space G/K equipped with a G-invariant Riemannian metric is uniformly $\mathcal{F}in$-contractible.*

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[1]These references only consider connected Lie groups. If G_0 is the component of the identity, then $G_0/G_0 \cap K \cong G/K$ and the general case follows.

It is easy to see that G/K is uniformly contractible, since G/K is contractible and has a transitive action by isometries. Thus, in the torsion-free case, the proof of Proposition 3.1 is trivial. For the general case, the proof depends on a fixed point result (Proposition 3.6), which was explained to the authors by Burkhard Wilking.

The following classical facts about Lie groups will be needed.

(3.2)   Let G be a Lie group with finitely many components that has a discrete central subgroup D such that the quotient G/D is compact. Then G is isomorphic to the semidirect product, $V \rtimes K$, of a vector group [2], V, with a compact group, K, acting on V. This can, for example, be extracted from the proof of Lemma XV.3.3 in [Hoc65] (see the end of the first paragraph on p. 183).

(3.3)   Let G be a Lie group, and let V be a normal vector subgroup of G such that G/V is compact. Then G is isomorphic to the semidirect product $V \rtimes G/V$. This follows from [Hoc65, Theorem III.2.3].

(3.4)   Let G be a semisimple Lie group with a finite number of components. Let K be a maximal compact subgroup of $\mathrm{Ad}_G(G)$, and let L be the preimage of K in G. Then any G-invariant metric on G/L has nonpositive sectional curvature (see [Hel62] [3].)

We make the following convenient definition.

Definition 3.5. Let H be a closed subgroup of a Lie group G. Assume that there is a G-invariant metric on G/H. The pair (G, H) has *nearby $\mathcal{F}in$-fixed points* if the following holds:

Let F be a finite subgroup of G and $R > 0$. Then there exists a $T > 0$ such that for every F-invariant subset B of G/H whose diameter is bounded by R, there is a point fixed by F such that the ball of radius T around this fixed point contains B.

It is not difficult to check that the existence of T, in the above definition, does not depend on the chosen metric, provided the metric is G-invariant.

Proposition 3.6. *Let G be a Lie group with finitely many components and K a maximal compact subgroup of G. Then (G, K) has nearby $\mathcal{F}in$-fixed points.*

The proof of Proposition 3.6 will require some preparations.

A Riemannian submersion $f \colon M \to N$ between Riemannian manifolds is a differentiable surjective map such that for every $x \in M$, the restriction of $df \colon T_x M \to T_{f(x)} N$ to the orthogonal complement of its kernel is an isometry. Important for us is the following property:

(3.7)   If ω is a smooth path in N and x is a lift of its the initial point to M, then there is a canonical lift of ω to M that begins at x and has the same arc length as ω.

Lemma 3.8. *Let $K \subset H$ be closed subgroups of the Lie group G.*

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[2]Lie groups isomorphic to $\mathbb{R}^n$ are called vector groups.

[3]This is well known. Since we did not find the statement in this form, we give a proof with precise references: We may assume that G is connected, since $G_0/G_0 \cap L \cong G/L$. Let $\mathbf{l} \oplus V$ be a Cartan decomposition of the Lie algebra $\mathbf{g}$ of G (see [Hel62, III.§7]). Then $\mathbf{l}$ is a maximal compactly imbedded subalgebra of $\mathbf{g}$ (see [Hel62, Proposition III.7.4]). This means that the subgroup of $\mathrm{Ad}_G(G)$ corresponding to $\mathbf{l}$ is a maximal compact subgroup. Since all maximal compact subgroups are conjugated [Hoc65, Theorem XV.3.1.], we may assume that this subgroup is K. Thus, the pair (G, L) is of noncompact type [Hel62, p.194/195], and any G-invariant metric on G/L has nonpositive curvature.

(i) Suppose that there are G-invariant metrics on G/K and G/H such that the projection $G/K \to G/H$ is a Riemannian submersion. If (H, K) and (G, H) have nearby $\mathcal{F}in$-fixed points, then so does (G, K).

(ii) If $\mathrm{Ad}_G(H)$ is compact, then there are G-invariant metrics on G/K and G/H such that the projection $G/K \to G/H$ is a Riemannian submersion.

Proof. (i) Let F be a finite subgroup of G, and let B be an F-invariant subset of G/K of diameter bounded by R. Since (G, H) has nearby $\mathcal{F}in$-fixed points and π is non-expanding, there is a fixed point gH in G/H such that $\pi(B)$ is contained in gH^T, the ball of radius T around gH. Here, T only depends on F and R (and not on B). For b in B, choose a geodesic, ω_b, in G/H from $\pi(b)$ to gH. Let $\varphi(b)$ be the endpoint of the lift of ω_b to G/K with initial point b. Let $C = \{\varphi(b) \mid b \in B\} \subset \pi^{-1}(gH)$. Property (3.7) implies $B \subset C^T$ and $C \subset B^T$. It follows that the diameter of C with respect to the metric on G/H is bounded by $R + 2T$. The diameter with respect to the restriction of the Riemannian metric to $\pi^{-1}(gH)$ may be larger, but will still be bounded by some number R' depending only on $R + 2T$. This is true because $H^g = g^{-1}Hg$ acts transitively and isometrically with respect to either distance on $\pi^{-1}(gH)$. Since (H, K) has nearby $\mathcal{F}in$-fixed points, there is a fixed point, hK, for $F^g = g^{-1}Fg$, and a $T' > 0$, depending only on R' and F^g, such that $g^{-1}(C) \subset \pi^{-1}(eH) = H/K$ is contained in $hK^{T'}$. Therefore, ghK is a fixed point for F, and B is contained in $ghK^{T+T'}$.

(ii) Since $\mathrm{Ad}_G(H)$ is compact, there is an H-invariant inner product on the Lie algebra $\mathbf{g}$ of G. The projection $G \to G/H$ induces an isomorphism from $\mathbf{h}^\perp$ to the tangent space $T_{eH}(G/H)$ of G/H at eH, where $\mathbf{h}$ is the Lie subalgebra of $\mathbf{g}$ corresponding to H. We use this isomorphism to transport the inner product from $\mathbf{h}^\perp$ to $T_{eH}(G/H)$. Since the inner product on $\mathbf{h}^\perp$ is H-invariant, this inner product extends to a G-invariant Riemannian metric on G/H. (Here, it is important that the inner product on $\mathbf{g}$ is H-invariant, otherwise the inner product on $T_{gH}(G/H)$ would depend on the choice of g, and not just on gH.) Similarly, we obtain a G-invariant metric on G/K. Thus, the projection, $\pi\colon G/K \to G/H$, is a Riemannian submersion. $\square$

Proof of Proposition 3.6. We begin by considering two special cases.

Case 1: If G is the semidirect product, $V \rtimes K$, of a vector group V with a compact group K, then $V \cong G/K$. That is, G/K is Euclidean space and the result follows from the Cartan Fixed Point Theorem.

Case 2: Suppose that G semisimple. Let L be the preimage under Ad_G of a maximal compact subgroup of $\mathrm{Ad}_G(G)$ that contains $\mathrm{Ad}_G(K)$. By Lemma 3.8 (ii), there exists a G-invariant metric on G/L, which is nonpositively curved by (3.4). By the Cartan Fixed Point Theorem, (G, L) has nearby $\mathcal{F}in$-fixed points. Since G/L is connected, L will only have a finite number of components. Since G is semisimple, the kernel of Ad_G is a discrete subgroup and the intersection of the kernel of Ad_G with the identity component G_0 of G, which we call D, is the center of G_0 [Hel62, Corollaries II.5.2+II.6.2]. Since $L \cap G_0/D$ is compact, (3.2) implies that $L \cap G_0$ is a semidirect product of a vector group group with a compact group. Therefore $L \cap G_0/K \cap G_0 \cong L/K$ is Euclidean space. Thus, (L, K) has nearby $\mathcal{F}in$-fixed points by the Cartan Fixed Point Theorem. By Lemma 3.8, (G, K) has nearby $\mathcal{F}in$-fixed points.

For the general case, we proceed by induction on $\dim G$. If G is not semisimple, there is a closed nontrivial normal subgroup that is either a vector group V or a torus T [Hoc65, Lemma XV.3.6]. If the subgroup in question is a torus, then we may assume that it is contained in K. Thus, the result for G/T implies it for G. If it is a vector group V, then consider the subgroup VK of G. By (3.3), the first case applies to VK. Of course, $G/VK \cong (G/V)/(VK/V)$. By induction, (VK, K) and (G, VK) have nearby $\mathcal{F}in$-fixed points. Choose a K-invariant inner product on the Lie algebra $\mathbf{g}$ of G. Denote by $\mathbf{v}$ the ideal of $\mathbf{g}$ corresponding to V. The projection $G \to G/V$ induces an isomorphism of the Lie algebra of G/V with $\mathbf{v}^{\perp}$ as a K-module. As in the proof of Lemma 3.8 (ii), we can use left translation to obtain G-invariant, resp. G/V-invariant, metrics on G/K, resp. $(G/V)/(VK/K)$, such that the projection $G/K \to (G/V)/(VK/K)$ is a Riemannian submersion. Using $G/VK \cong (G/V)/(VK/V)$ and Lemma 3.8 (i), it follows that (G/K) has nearby $\mathcal{F}in$-fixed points. $\square$

Lemma 3.9. *Let G be a Lie group with finitely many components and K a maximal compact subgroup. Equip G/K with a G-invariant Riemannian metric. Then for every $T > 0$ there is an $S > 0$ such that the ball of radius T around eK is K-equivariantly contractible inside the ball of radius S.*

Proof. By [Hoc65, Theorem XV.3.1], there is a finite dimensional vector space V with a linear K-action and a K-equivariant homeomorphism $G/K \to V$ sending eK to 0. Thus, eK^T is K-equivariantly contractible in G/K. By compactness, this contraction happens inside some ball of finite radius. $\square$

Proof of Proposition 3.1. Let F be a finite subgroup of G. Let $R > 0$ be given. By Proposition 3.6, there is a $T > 0$ such that for every F-invariant subset B of G/K whose diameter is bounded by R, there is a point fixed by F such that the ball of radius T around this fixed point contains B. By Lemma 3.9, there is an $S > 0$ such that eK^T is K-equivariantly contractible inside the ball of radius S.

Let B' an F-invariant subset of G/K whose diameter is bounded by R. Then there is an F-fixed point gK in G/K such that B' is contained in gK^T. Since gK is an F-fixed point, $F^g = g^{-1}Fg$ is a subgroup of K. Thus, $g^{-1}B'$ is contained in eK^T and is F^g-invariant. Therefore, $g^{-1}B'$ is F^g-equivariantly contractible in eK^S. Applying g, this means that B' is F-equivariantly contractible in gK^S. $\square$

4. Open covers and simplicial compexes

A map $f \colon X \to Y$ between metric spaces is *metrically coarse* if it is proper and satisfies the following growth condition: for all $R > 0$ there is an $S > 0$ such that

$$d_X(x, y) < R \quad \Longrightarrow \quad d_Y(f(x), f(y)) < S.$$

Two such maps, f and g, are said to be *bornotopic* if there is a constant $C > 0$ such that $d_Y(f(x), g(x)) < C$ for all x in X (see [HR95, Section 2]). A metrically coarse homotopy between proper continuous maps is called a *metric homotopy*. In particular, a metric homotopy is a bornotopy and a proper homotopy.

For the following lemma compare [HR95, Lemma 3.3].

Lemma 4.1. *Let X be a proper metric space that has a finite dimensional CW-structure such that the diameters of its cells are uniformly bounded, and let Y be a uniformly contractible proper metric space. Let A be a subcomplex of X containing*

the 0-cells of X. Then every continuous metrically coarse map $f_0 \colon A \to Y$ can be extended to a continuous metrically coarse map $f \colon X \to Y$.

Proof. Let X_n be the union of A with the n-skeleton of X. Since X_n is obtained from X_{n-1} by attaching n-cells, we extend f_0 inductively to $f_n \colon X_n \to Y$. For every n-cell, e, not in A, we must extend to all of e the restriction of f_{n-1} to the boundary, ∂e, of e. Because f_{n-1} is metrically coarse and cells in X have uniformly bounded diameter, $f_{n-1}(\partial e)$ has uniformly bounded diameter. Since Y is uniformly contractible, there is a continuous metrically coarse extension of f_{n-1} to X_n. This finishes the construction of f.

It remains to show that f is proper. Since the diameters of cells in X are uniformly bounded and X is finite dimensional, there is an $R > 0$ such that $X = (X^0)^R$, where X^0 denotes the 0-skeleton of X. Since X is a proper metric space, this implies that a subset B of X has compact closure if and only if $B^R \cap X^0$ is finite. Let B_α be a closed ball of radius α in Y. Because f is metrically coarse, there is an $S > 0$ such that $(f^{-1}(B_\alpha))^R$ is contained in $f^{-1}((B_\alpha)^S)$. Since $(B_\alpha)^S$ is compact, f_0 is proper, and X^0 is closed in X, it follows that $(f^{-1}(B_\alpha))^R \cap X^0 \subset f^{-1}((B_\alpha)^S) \cap X^0$ is finite. Since $f^{-1}(B_\alpha)$ is closed, this implies that $f^{-1}(B_\alpha)$ is compact. Therefore, f is proper. $\qquad\square$

If $\mathcal{U}$ is an open cover of a space X, then the realization of its nerve, $|\mathcal{U}|$, is a simplicial complex. Denote by $[U]$ the vertex of $|\mathcal{U}|$ corresponding to U in $\mathcal{U}$. A partition of unity subordinate to $\mathcal{U}$, $(\varphi_U)_{U \in \mathcal{U}}$, induces a map $g \colon X \to |\mathcal{U}|$ defined by:

$$(4.2) \qquad g(x) = \sum_{U \in \mathcal{U}} \frac{\varphi_U(x)[U]}{\sum_{V \in \mathcal{U}} \varphi_V(x)}$$

The *Euclidean path length metric* [4] on a simplicial complex is the unique path length metric that restricts to the standard Euclidean metric on each simplex. For an open cover $\mathcal{U}$, we will always equip $|\mathcal{U}|$ with the Euclidean path length metric.

Proposition 4.3. *Let X be a complete proper path metric space and let $\mathcal{U}$ be a locally finite, bounded, finite dimensional cover of X whose Lebesgue number is positive. If $g \colon X \to |\mathcal{U}|$ is induced by a partition of unity subordinate to $\mathcal{U}$ as in (4.2), then g is a bornotopy equivalence.*

Proof. This follows from [Roe91, Section 3] (see also [HR95, Proposition 3.2]). In these references the spherical metric rather than the Euclidean metric is used, but since $|\mathcal{U}|$ is finite dimensional, this distinction is not important. $\qquad\square$

Lemma 4.4. *Let X be a uniformly contractible complete proper path metric space. Assume that X has the structure of a finite dimensional CW-complex. Let $\mathcal{U}$ be a locally finite, bounded, finite dimensional cover of X whose Lebesgue number is positive, and let $g \colon X \to |\mathcal{U}|$ be induced by a partition of unity subordinate to $\mathcal{U}$ as in (4.2). Then g has a right homotopy inverse up to metric homotopy. That is, there is a continuous and metrically coarse map $f \colon |\mathcal{U}| \to X$ and a metric homotopy $H \colon X \times [0,1] \to X$ from $f \circ g$ to id_X.*

---

[4]In [Yu98] and [Bar03b] the spherical metric has been used. Using the Euclidean metric is convenient for computation in Proposition 4.6. In any event, the difference between the Euclidean and the spherical metric is not important for this paper.

Proof. By refining the CW-structure if necessary, we can assume that the cells in X have uniformly bounded diameter. By Proposition 4.3, g has a bornotopy inverse f. Using Lemma 4.1, we can assume that f is also continuous. The existence of H follows by applying Lemma 4.1 to the subspace $X \times \{0, 1\} \subset X \times [0, 1]$ and the map that is $f \circ g$ on $X \times \{0\}$ and the identity of X on $X \times \{1\}$. $\qquad\square$

Although the Euclidean path length metric changes under restriction to subcomplexes, there is the following estimate.

Lemma 4.5. *Let σ and τ be intersecting faces of the k-simplex Δ. Denote by d_{pl} the Euclidean path length metric on the subcomplex of Δ spanned by σ and τ, and denote by d_Δ the Euclidean standard metric on Δ. Then for x in σ and y in τ*

$$d_{pl}(x, y) \leq 3\sqrt{k+1} \cdot d_\Delta(x, y).$$

Proof. Let v_0 be a vertex that is contained in $\sigma \cap \tau$. For a face ρ of Δ that contains v_0, define the projection $p_\rho \colon \Delta \to \rho$ as the simplicial map which is the identity on ρ and maps all vertices not in ρ to v_0. Using the Cauchy-Schwarz inequality,

$$d_\Delta(p_\rho(z), p_\rho(z')) \leq \sqrt{k+1} \cdot d_\Delta(z, z') \quad \forall z, z' \in \Delta.$$

Note that $p_\sigma(x) = x$, $p_\tau(y) = y$, $p_{\sigma \cap \tau}(x) = p_\tau(x)$ and $p_{\sigma \cap \tau}(y) = p_\sigma(y)$. Therefore,

$$
\begin{aligned}
d_{pl}(x, y) &\leq d_{pl}(x, p_{\sigma \cap \tau}(y)) + d_{pl}(p_{\sigma \cap \tau}(y), p_{\sigma \cap \tau}(x)) + d_{pl}(p_{\sigma \cap \tau}(x), y) \\
&= d_{pl}(p_\sigma(x), p_\sigma(y)) + d_{pl}(p_{\sigma \cap \tau}(y), p_{\sigma \cap \tau}(x)) + d_{pl}(p_\tau(x), p_\tau(y)) \\
&= d_\Delta(p_\sigma(x), p_\sigma(y)) + d_\Delta(p_{\sigma \cap \tau}(y), p_{\sigma \cap \tau}(x)) + d_\Delta(p_\tau(x), p_\tau(y)) \\
&\leq 3\sqrt{k} \cdot d_\Delta(x, y).
\end{aligned}
$$

$\qquad\square$

The finite asymptotic dimension of a metric space is equivalent to the existence of certain contracting maps to finite dimensional simplicial complexes (see [Gro93, p.30]). We will use the following version of this (compare [Yu98, Lemma 6.3]).

Lemma 4.6. *Let $\mathcal{U}$ be an n-dimensional bounded cover of a path length metric space X. Assume that the Lebesgue number R of $\mathcal{U}$ is positive. Then there is a partition of unity subordinate to $\mathcal{U}$ such that for the induced map $g \colon X \to |\mathcal{U}|$ (4.2),*

$$d_{pl}(g(x), g(y)) \leq C_n \frac{d(x, y)}{R}$$

for all x, $y \in X$. Here, d_{pl} denotes the Euclidean path length metric on $|\mathcal{U}|$, and C_n is a constant that depends only on n.

Proof. Let $V_{\mathcal{U}}$ be the vector space of sequences of real numbers indexed by $\mathcal{U}$. There is a canonical embedding $|\mathcal{U}| \to V_{\mathcal{U}}$. We will use both the l^1-norm, $\|\cdot\|_1$, and the l^2-norm, $\|\cdot\|_2$, on $V_{\mathcal{U}}$. For x in X and U in $\mathcal{U}$, let $x_U = d(x, X - U)$ and $f(x) = (x_U)_{U \in \mathcal{U}} \in V_{\mathcal{U}}$. Define a partition of unity subordinate to $\mathcal{U}$ by $\varphi_U(x) = \frac{x_U}{\|f(x)\|_1}$. The map $g \colon X \to |\mathcal{U}| \subset V_{\mathcal{U}}$ induced by this partition of unity is given by $x \mapsto \frac{f(x)}{\|f(x)\|_1}$.

Let x, $y \in X$ be given. Clearly, $\|f(x)\|_2 \leq \|f(x)\|_1$. Because R is the Lebesgue number of $\mathcal{U}$, there is at least one U in $\mathcal{U}$ for which $y_U \geq R$. Therefore, $\|f(y)\|_1 \geq R$. Since $\mathcal{U}$ is n-dimensional, there are at most $2(n+1)$ members of $\mathcal{U}$ for which $x_U \neq 0$ or $y_U \neq 0$. From $|x_U - y_U| \leq d(x, y)$, we conclude $\big| \|f(y)\|_1 - \|f(x)\|_1 \big| \leq$

$2(n+1)\,d(x,y)$ and $\|f(x)-f(y)\|_2 \leq \sqrt{2(n+1)}\,d(x,y) \leq 2(n+1)\,d(x,y)$. Using these estimates, it follows that

$$
\begin{aligned}
\|g(x)-g(y)\|_2 &= \left\| \frac{f(x)}{\|f(x)\|_1} - \frac{f(y)}{\|f(y)\|_1} \right\|_2 \\
&\leq \left\| \frac{\|f(y)\|_1 f(x) - \|f(x)\|_1 f(x)}{\|f(x)\|_1 \|f(y)\|_1} \right\|_2 + \left\| \frac{\|f(x)\|_1 f(x) - \|f(x)\|_1 f(y)}{\|f(x)\|_1 \|f(y)\|_1} \right\|_2 \\
&\leq \frac{\big|\|f(y)\|_1 - \|f(x)\|_1\big| \|f(x)\|_2}{\|f(x)\|_1 \|f(y)\|_1} + \frac{\|f(x)-f(y)\|_2}{\|f(y)\|_1} \\
&\leq \frac{4(n+1)d(x,y)}{R}.
\end{aligned}
$$

Restricted to simplices, the embedding $|\mathcal{U}| \to V_\mathcal{U}$ is an isometry (using the l^2-norm on $V_\mathcal{U}$). If $d(x,y) < R$, then there is a U in $\mathcal{U}$ containing x and y, since R is the Lebesgue number of $\mathcal{U}$. Thus, there are simplices σ and τ of $|\mathcal{U}|$ such that $g(x) \in \sigma$, $g(y) \in \tau$, and $[U] \in \sigma \cap \tau$. In particular, $\sigma \cap \tau$ is not empty. Because σ and τ have dimension less or equal to n, they span a simplex Δ in $V_\mathcal{U}$ whose dimension is at most $2n$. Since the l^2-norm gives the Euclidean metric on Δ, we can use Lemma 4.5 to estimate

$$
d_{pl}(g(x),g(y)) \leq 3\sqrt{2n+1} \cdot \|g(x)-g(y)\|_2 \leq 12\sqrt{2n+1}(n+1)\frac{d(x,y)}{R},
$$

whenever $d(x,y) \leq R$. Since the metric on X is a path length metric, the inequality follows for all $x,\,y \in X$. $\qquad\square$

5. Controlled algebra

Let X be a proper metric space and $\mathcal{A}$ a small additive category. Let Γ be a group that acts on X by isometries and on $\mathcal{A}$ by additive functors. (We will consider only left actions.) Fundamental to this paper will be the additive category $\mathcal{A}(X)$ of certain continuously controlled modules. This is a minor variation of similar constructions from the literature, the first of which appeared in [ACFP94].

Let $Z = X \times \Gamma \times [0,1)$. An object, M, in $\mathcal{A}(X)$ is given by a sequence of objects $(M_z)_{z \in Z}$ in $\mathcal{A}$, subject to the conditions:

(5.1)   The image of $\operatorname{supp} M = \big\{ z \,\big|\, M_z \neq 0 \big\}$ under the projection $Z \to X \times [0,1)$ is locally finite.

(5.2)   For every $x \in X$ and $t \in [0,1)$, $\operatorname{supp} M \cap \big(\{x\} \times \Gamma \times \{t\}\big)$ is finite.

A morphism, $\varphi \colon M \to N$, in $\mathcal{A}(X)$ is given by a sequence of morphisms, $(\varphi_{z,z'} \colon M_{z'} \to M_z)_{(z,z') \in Z^2}$, in $\mathcal{A}$, subject to the conditions:

(5.3) $\operatorname{supp} \varphi = \big\{ (z,z') \,\big|\, \varphi_{z,z'} \neq 0 \big\}$ is *continuously controlled* at $X \times \{1\}$. That is, for every $x \in X$ and every open neighborhood U of $(x,1)$ in $X \times [0,1]$, there is a (smaller) open neighborhood V of $(x,1)$ in $X \times [0,1]$ such that $(X \times [0,1) - U) \times V$ and $V \times (X \times [0,1) - U)$ do not intersect the image of $\operatorname{supp} \varphi$ under the projection $Z \to X \times [0,1]$.

(5.4)   For a fixed z in Z, $\big\{ z' \,\big|\, (z,z') \in \operatorname{supp} \varphi$ or $(z',z) \in \operatorname{supp} \varphi \big\}$ is finite.

(5.5)   There exists an $R > 0$ such that $((x,\gamma,t),(x',\gamma',t')) \in \operatorname{supp} \varphi$ implies $d(x,x') < R$.

Composition of morphisms is given by the usual matrix multiplication.

If $\mathcal{X}$ is a collection of subsets of Z, then we define $\mathcal{A}(\mathcal{X})$ as the full subcategory of $\mathcal{A}(X)$ whose objects, M, satisfy the additional condition:

(5.6)   There is an $S \in \mathcal{X}$ such that $\operatorname{supp} M \subset S$.

If $\mathcal{X}$ is closed under finite unions, then $\mathcal{A}(\mathcal{X})$ is again an additive category. If $\mathcal{Y}$ is another collection of subsets of Z such that for every $S \in \mathcal{Y}$ there is a $T \in \mathcal{X}$ such that $S \subset T$, then $\mathcal{A}(\mathcal{Y})$ is a subcategory of $\mathcal{A}(\mathcal{X})$. Furthermore, $\mathcal{A}(\mathcal{Y})$ will define a Karoubi filtration [CP95, Definition 1.27] of $\mathcal{A}(\mathcal{X})$ if, in addition, the following is satisfied:

(5.7)   For every $S \in \mathcal{Y}$ and morphism φ in $\mathcal{A}(\mathcal{X})$, there is a $T \in \mathcal{Y}$ such that $S^\varphi = \left\{ z \mid \exists\, z' \in S \text{ such that } \varphi_{z,z'} \neq 0 \right\}$ is contained in T.

The quotient of this Karoubi filtration will be denoted by $\mathcal{A}(\mathcal{X}, \mathcal{Y})$.

Clearly, $\mathcal{A}(X) = \mathcal{A}(\{X\})$ and $\mathcal{A}(\mathcal{X}) = \mathcal{A}(\mathcal{X}, \emptyset)$. If $\mathcal{X}$ and $\mathcal{Y}$ are Γ-invariant, then the formula $(g(M))_z = g(M_{g^{-1}z})$ defines an action of Γ on $\mathcal{A}(\mathcal{X}, \mathcal{Y})$. For a subgroup G of Γ, denote the corresponding fixed point category by $\mathcal{A}^G(\mathcal{X}, \mathcal{Y})$. It is not difficult to check that taking Karoubi quotients and taking fixed categories commute. Therefore, $\mathcal{A}^G(\mathcal{X}, \mathcal{Y})$ is the quotient of $\mathcal{A}^G(\mathcal{X})$ by $\mathcal{A}^G(\mathcal{Y})$.

Let $p \colon X \to X'$ be a continuous map. For a closed subset Y of X, let $\mathcal{X}(Y, p)$ be the collection of subsets S of Z with the following properties:

(5.8)   If $(x, 1)$ is a limit point of the image of S under the projection $Z \to X \times [0, 1)$, then $x \in Y$.

(5.9)   There is an $R > 0$ such that the image of S under the projection $Z \to X$ is contained in Y^R.

(5.10)   There is a compact set $K_0 \subset X'$ such that the image of S under the composition of the projection $Z \to X$ with p is contained in K_0.

Let $\mathcal{X}(Y, p)_0$ be the collection of subsets S of Z that satisfy (5.9), (5.10) and the following strengthening of (5.8):

(5.11)   The set of limit points of the image of S under the projection $Z \to X \times [0, 1)$ is disjoint from $X \times \{1\}$.

Let G be a subgroup of Γ such that Y is G-invariant. The controlled categories that we will use are:

$$\mathcal{A}_p^G(Y) = \mathcal{A}^G(\mathcal{X}(Y, p)),$$
$$\mathcal{A}_p^G(Y)_0 = \mathcal{A}^G(\mathcal{X}(Y, p)_0),$$
$$\mathcal{A}_p^G(Y)^\infty = \mathcal{A}^G(\mathcal{X}(Y, p), \mathcal{X}(Y, p)_0),$$
$$\mathcal{A}_p^G(X, Y) = \mathcal{A}^G(\mathcal{X}(X, p), \mathcal{X}(Y, p)).$$

Note that the definitions of these categories depend on the metric space X, even though this is not reflected in the notation. However, because X is a proper metric space, changing X to a smaller or larger proper metric space that still contains Y will only change the categories up to G-equivariant equivalence. Since all of our metric spaces will be proper metric spaces, we can disregard this dependence on X. It should also be noted that condition (5.5) is only important in the definitions of $\mathcal{A}_p^G(Y)$ and $\mathcal{A}_p^G(Y)_0$ and not in the definition of $\mathcal{A}_p^G(Y)^\infty$, where it affects the category only up to equivalence (compare [Bar03b, Lemma 3.15]). Similarly, our notation is slightly imprecise because even before taking fixed categories these categories depend on the group we have in mind. But this dependence also only changes the category up to equivariant equivalence and can safely be ignored.

Let $\mathbb{K}^{-\infty}$ denote the functor from the category of small additive categories to the category of spectra, which assigns an additive category to its associated non-connective K-theory spectrum [PW85]. A crucial fact is that applying $\mathbb{K}^{-\infty}$ to a Karoubi filtration produces a fibration of spectra, which induces a long exact sequence in K-theory [CP95, Theorem 1.28]. By definition, the two following sequences are Karoubi sequences. That is, the final category is a Karoubi quotient of the middle category by the first. Therefore, each induces a long exact sequence in K-theory.

(5.12) $\mathcal{A}_p^G(X)_0 \to \mathcal{A}_p^G(X) \to \mathcal{A}_p^G(X)^\infty$

(5.13) $\mathcal{A}_p^G(Y) \to \mathcal{A}_p^G(X) \to \mathcal{A}_p^G(X,Y)$

If, in addition, $Y^R = X$ for some $R > 0$, then

(5.14) $\mathcal{A}_p^G(Y)^\infty \to \mathcal{A}_p^G(X)^\infty \to \mathcal{A}_p^G(X,Y)$

is also Karoubi sequence. To see this, consider the following commutative diagram in which the first two columns and the second two rows are Karoubi filtrations.

$$
\begin{array}{ccc}
\mathcal{A}_p^G(Y)_0 & \xrightarrow{\cong} & \mathcal{A}_p^G(X)_0 \\
\downarrow & & \downarrow \\
\mathcal{A}_p^G(Y) & \longrightarrow \mathcal{A}_p^G(X) \longrightarrow & \mathcal{A}_p^G(X,Y) \\
{\scriptstyle a}\downarrow & {\scriptstyle b}\downarrow & {\scriptstyle c}\downarrow \\
\mathcal{A}_p^G(Y)^\infty & \longrightarrow \mathcal{A}_p^G(X)^\infty \longrightarrow & Q
\end{array}
$$

The assumption $Y^R = X$ implies that $\mathcal{A}_p^G(Y)_0 \to \mathcal{A}_p^G(X)_0$ is an equivalence of categories. A short exercise in the definition of Karoubi filtrations shows that c is an equivalence of categories.

In [Wei02], it is proven that $X \mapsto K_*(\mathcal{A}_{\mathrm{id}_X}(X)^\infty)$ is a locally finite homology theory. In particular, it is homotopy invariant. It follows from the definition that $\mathcal{A}_{\mathrm{id}_X}(X)_0$ is functorial in X for metrically coarse maps and that bornotopic maps induce the same map in K-theory. These facts imply:

(5.15)   The category $\mathcal{A}(X)$ is functorial in X for metrically coarse continuous maps (compare [Bar03b, Remarks 3.5, 3.10]). If $f, g\colon X \to Y$ are two such maps that are metrically homotopic (see Section 4), then f and g induce the same map from $K_*(\mathcal{A}(X))$ to $K_*(\mathcal{A}(Y))$.

In [BR05, Definition 2.1], the additive category $\mathcal{A} *_\Gamma T$ is constructed for a Γ-set T. By [BR05, Proposition 2.8(iii)], there is a canonical equivalence of categories $\mathcal{A} *_H H/H \to \mathcal{A} *_G G/H$, for every subgroup H of Γ. If $\mathcal{A}$ is the category of finitely generated free R-modules for a ring R and the action of H on $\mathcal{A}$ is trivial, then $\mathcal{A} *_H H/H$ is equivalent to the category of finitely generated free $R[H]$-modules. For this reason, we will denote the category $\mathcal{A} *_H H/H$ by $\mathcal{A}[H]$ for an arbitrary $\mathcal{A}$, and call the objects of this category $\mathcal{A}[H]$-modules. In [BR05, Section 3], the $\mathrm{Or}(\Gamma)$-spectrum $\mathbf{K}_\mathcal{A}$ is also introduced (using the $\mathbb{K}^{-\infty}$-functor). Associated to it is the assembly map

(5.16) $H_*(\underline{E}\Gamma; \mathbf{K}_\mathcal{A}) \to H_*(\mathrm{pt}; \mathbf{K}_\mathcal{A}) = K_*(\mathcal{A}[\Gamma])$.

If $\mathcal{A}$ is the category of finitely generated free R-modules, then this is the assembly map (0.1). Let $p_\Gamma\colon \underline{E}\Gamma \to \underline{E}\Gamma/\Gamma$ denote the quotient map. It is not hard to check

that $\mathcal{A}[\Gamma]$ is equivalent to the category $\mathcal{A}^{\Gamma}_{p_\Gamma}(\underline{E}\Gamma)_0$. Controlled algebra can be used to describe assembly maps as forget control maps. An instance of this is [HP04], where the continuously controlled category $\mathcal{B}_\Gamma(\underline{E}\Gamma\times[0,1);R)$ is used to identify various versions of the assembly map. If one modifies the definition of this category to allow for coefficients in $\mathcal{A}$, then, since Γ acts on $\underline{E}\Gamma$ with finite isotropy, the fixed point category $\mathcal{A}^{\Gamma}_{p_\Gamma}(\underline{E}\Gamma)^\infty$ can be identified with the continuously controlled category $\mathcal{B}_\Gamma(\underline{E}\Gamma\times[0,1);\mathcal{A})^{>0}$. The only difference between the two categories is condition (5.5), but as mentioned before, this changes the category only up to equivalence. Therefore, [HP04, Theorem 7.4] implies the following fact.

(5.17)   If $\underline{E}\Gamma$ is equipped with a Γ-invariant metric, then the boundary map in the long exact sequence associated to (5.12), with $X = \underline{E}\Gamma$, is equivalent to the assembly map (5.16).

6. A vanishing result

In this section, a key component of the proof of Theorem 8.1 is established, namely Proposition 6.2 below.

Proposition 6.1. *Let X be a proper metric space. For each $n \in \mathbb{N}$, let Q_n be a simplicial complex equipped with the Euclidean path length metric and $g_n\colon X \to Q_n$ a continuous and metrically coarse map satisfying the following: For every $R > 0$ there exists an $S > 0$ such that for all x and y in X with $d(x,y) < R$, $d(g_n(x), g_n(y)) < \frac{S}{n}$. Then for any $a \in K_m(\mathcal{A}(X))$ there is an $n_0 = n_0(a)$ such that $(g_n)_*(a) = 0 \in K_m(\mathcal{A}(Q_n))$ for all $n \geq n_0$.*

Proof. This is almost [Bar03b, Corollary 4.3]. In [Bar03b], the spherical metric rather then the Euclidean metric is used, but this does not affect the argument. It is only important that the metric is the same on every simplex.   □

Proposition 6.2. *Let X be a uniformly contractible, complete, proper, path length metric space of finite asymptotic dimension. Assume that X has the structure of a finite dimensional CW-complex. Let $\mathcal{X}$ be a collection of closed subsets, S, of $X\times[0,1)$, where $S = K\times[0,1)$ for some closed subset K of X, that is closed under finite unions, and assume that for every closed subset K of X and every $\alpha > 0$, $K^\alpha\times[0,1)$ is contained in $\mathcal{X}$. Then the K-theory of $\mathcal{A}(\mathcal{X})$ vanishes.*

Proof. Since X has finite asymptotic dimension, there is a sequence of bounded open covers $\mathcal{U}_n$ of X such that the Lebesgue number of $\mathcal{U}_n$ exceeds n. By Lemma 2.1, we can assume that each $\mathcal{U}_n$ is locally finite. By Lemma 4.6 and Proposition 6.1, there is a sequence of maps, $g_n\colon X \to |\mathcal{U}_n|$, induced by partitions of unity, such that for every closed subset Y of X and every $a \in K_*(\mathcal{A}(Y))$, there is an n_0 such that $(g_n)_*(a) = 0$ for $n \geq n_0$. By Lemma 4.4, each g_n is invertible up to metric homotopy. Thus, for each n, there is a continuous metrically coarse map $f_n\colon |\mathcal{U}_n| \to X$ and a metric homotopy $H_n\colon X\times[0,1] \to X$ from $f_n \circ g_n$ to id_X.

Let $b \in K_*(\mathcal{A}(\mathcal{X}))$ be given. Choose a closed subset K of X such that b is the image of some $a \in K_*(\mathcal{A}(K))$ under the inclusion $\iota_K\colon \mathcal{A}(K) \to \mathcal{A}(\mathcal{X})$. Let Q_n be the smallest subcomplex of $|\mathcal{U}_n|$ that contains $g_n(K)$. Since f_n and H_n are metrically coarse, there is an $\alpha_n > 0$ such that $f_n(Q_n)$, $H_n(K\times[0,1]) \subset K^{\alpha_n}$. Consider the restriction maps $g_n|_K\colon K \to Q_n$ and $f_n|_{Q_n}\colon Q_n \to K^{\alpha_n}$. Then the restriction of H_n to $K\times[0,1]$ gives a metric homotopy from $f_n|_{Q_n} \circ g_n|_K$ to the inclusion $i_{\alpha_n}\colon K \to K^{\alpha_n}$. From (5.15) we conclude that $(f_n|_{Q_n} \circ g_n|_K)_*(a) =$

$(i_{\alpha_n})_*(a)$. By Proposition 6.1, eventually $(g_n|_K)_*(a) = 0$. The conclusion now follows since $i_{\alpha_n}(a)$ also maps to b under the inclusion $\iota_{K^{\alpha_n}} : \mathcal{A}(K^{\alpha_n}) \to \mathcal{A}(\mathcal{X})$. $\square$

7. The Descent Principle and Homotopy Fixed Sets

Let $\mathbf{S}$ be a spectrum with Γ-action. For a space X with Γ-action, let $\mathrm{Map}_\Gamma(X, \mathbf{S})$ be the spectrum whose n-th space is $\mathrm{Map}_\Gamma(X, \mathbf{S}_n)$, where $\mathbf{S}_n$ is the n-th space in the spectrum $\mathbf{S}$. The fixed point spectrum, $\mathbf{S}^\Gamma$, is defined to be $\mathrm{Map}_\Gamma(\mathrm{pt}, \mathbf{S})$. Let $\mathcal{F}$ be a family of subgroups of Γ that is closed under conjugation and taking subgroups. If X is a space with Γ-action, then its $\mathcal{F}$-homotopy fixed point set is defined by $X^{h_\mathcal{F}\Gamma} = \mathrm{Map}_\Gamma(E_\mathcal{F}\Gamma, X)$. In general, spectra can be difficult to work with, however, many arguments for spaces can be extended to Ω-spectra by applying them levelwise. A particularly useful property they possess is that the homotopy groups of a product of Ω-spectra is the product of the homotopy groups. In order to define the $\mathcal{F}$-homotopy fixed points of a spectrum with Γ-action, recall that there is a fibrant replacement functor, $\mathbf{R} : \mathrm{SPECTRA} \to \Omega - \mathrm{SPECTRA}$, that comes equipped with a natural weak equivalence $\mathbf{A} \to \mathbf{R}(\mathbf{A})$. For a construction of this functor, see for example, [LRV03, Section 2]. If $\mathbf{S}$ is a spectrum with Γ-action then so is $\mathbf{R}(\mathbf{S})$, and it is not difficult to check that $\mathbf{R}$ commutes with taking fixed points. That is, $(\mathbf{R}(\mathbf{S}))^\Gamma = \mathbf{R}(\mathbf{S}^\Gamma)$. The *$\mathcal{F}$-homotopy fixed point spectrum* [5] is defined to be the spectrum

$$\mathbf{S}^{h_\mathcal{F}\Gamma} = \mathrm{Map}_\Gamma(E_\mathcal{F}\Gamma, \mathbf{R}(\mathbf{S})).$$

Notice that the projection $E_\mathcal{F}\Gamma \to \mathrm{pt}$ induces a natural transformation $\mathbf{S}^\Gamma \to \mathbf{S}^{h_\mathcal{F}\Gamma}$.

Lemma 7.1. *Let $\mathcal{F}$ be a family of subgroups of Γ.*

    (i) If $F : \mathbf{S} \to \mathbf{T}$ is an equivariant map of spectra with Γ-action such that $F^G : \mathbf{S}^G \to \mathbf{T}^G$ is a weak homotopy equivalence for every subgroup G in $\mathcal{F}$, then $\mathbf{S}^{h_\mathcal{F}\Gamma} \simeq \mathbf{T}^{h_\mathcal{F}\Gamma}$.

    (ii) Let $\mathbf{B}$ be an Ω-spectrum with a G-action, where G is in $\mathcal{F}$, and let Γ act on $\mathbf{S} = \mathrm{Map}_G(\Gamma, B)$ by $(\gamma f)(x) = f(\gamma^{-1}x)$, where $\gamma \in \Gamma$. Then $\mathbf{S}^\Gamma \simeq \mathbf{S}^{h_\mathcal{F}\Gamma}$.

Proof. (i) The proof of [Ros04, Lemma 4.1] shows that the corresponding result holds for spaces, but this also implies the result for spectra. For every $G \in \mathcal{F}$ and $n \in \mathbb{N}$, $(\mathbf{R}(\mathbf{S})_n)^G \to (\mathbf{R}(\mathbf{T})_n)^G$ is a weak equivalence because $\mathbf{R}$ commutes with fixed points. Thus, $(\mathbf{R}(\mathbf{S})_n)^{h_\mathcal{F}\Gamma} \to (\mathbf{R}(\mathbf{T})_n)^{h_\mathcal{F}\Gamma}$ is weak equivalence for all $n \in \mathbb{N}$.

(ii) The proof of [Ros04, Lemma 4.2] shows that the corresponding result holds for spaces. The statement for spectra can be deduced from this as follows. A product of Ω-spectra is again an Ω-spectrum. In particular, $\mathbf{S}^H$ is an Ω-spectrum for every subgroup H of Γ. Thus, $(\mathbf{S}_n)^H \to (\mathbf{R}(\mathbf{S})_n)^H$ is a weak equivalence for every n. Therefore, $(\mathbf{S}_n)^{h_\mathcal{F}\Gamma} \to (\mathbf{R}(\mathbf{S})_n)^{h_\mathcal{F}\Gamma}$ is a weak equivalence for every n. On the other hand, $(\mathbf{S}_n)^\Gamma \to (\mathbf{S}_n)^{h_\mathcal{F}\Gamma}$ is a weak equivalence by the space version of (ii). $\square$

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[5]Both authors of this paper are guilty of stating an incorrect definition of the homotopy fixed point spectrum (forgetting $\mathbf{R}$) [Bar03a], [Ros04]. However, in those papers, homotopy fixed points are only applied to Ω-spectra. In this case, both definitions yield weakly equivalent spectra. Thus, the main results are not affected. We thank Holger Reich for pointing out that the correct definition of the homotopy fixed point spectrum must involve the functor $\mathbf{R}$.

Lemma 7.2. *Let $\mathcal{F}$ be the family of finite subgroups of Γ, H a finite subgroup of Γ, $\mathcal{B}$ an additive category with an H-action, and let Γ act on $\mathcal{C} = \prod_{\Gamma/H} \mathcal{B}$ as it does on the product in Lemma 7.1 (ii). Then*

$$\mathbb{K}^{-\infty}(\mathcal{C})^{\Gamma} \simeq \mathbb{K}^{-\infty}(\mathcal{C})^{h_{\mathcal{F}}\Gamma}.$$

Proof. For $\mathcal{B} = \mathcal{A}(\mathrm{pt})^{\infty}$ this is proven in [Ros04, Theorem 6.3] using Lemma 7.1 (ii) and the fact that $\mathbb{K}^{-\infty}$ commutes with infinite products [Car95]. The same proof works in the general case. $\qquad\square$

Remark 7.3. In order for the L-theory version of Lemma 7.2 to be true, it must also be assumed that for sufficiently large i, $K_{-i}(\mathcal{B}[H]) = 0$. This is needed because the compatibility of $\mathbb{L}^{-\infty}$ with infinite products is only known if the K-theory vanishes in degree $-i$ for sufficiently large i (see [CP95, p.756]).

Proposition 7.4, below, is an important fact needed to prove the Descent Principle (Theorem 7.5). Its proof is based on work of Carlsson and Pedersen [CP95, Theorem 2.11] who proved the result in the case where Γ is torsion-free and X is a finite Γ-CW complex (meaning that X has finitely many Γ-cells). This was later generalized by the second author [Ros04] to include groups with torsion. The result proven here relaxes the finiteness condition on X, requiring only that X be a finite dimensional Γ-CW complex.

Proposition 7.4. *Let Γ be a discrete group, $\mathcal{F}$ the family of finite subgroups of Γ, X a finite dimensional Γ-CW complex with finite isotropy, and p_{Γ} the quotient map $X \to X/\Gamma$. Then, for every Γ-invariant metric on X,*

$$\mathbb{K}^{-\infty}\big(\mathcal{A}_{p_{\Gamma}}(X)^{\infty}\big)^{\Gamma} \simeq \mathbb{K}^{-\infty}\big(\mathcal{A}_{p_{\Gamma}}(X)^{\infty}\big)^{h_{\mathcal{F}}\Gamma}.$$

Proof. Proceed by induction on the skeleta of X. Let X_0 denote the 0-skeleton of X. For some indexing set J, $X_0 = \coprod_{j \in J} \Gamma/H_j$, where $H_j \in \mathcal{F}$ for every $j \in J$. Since we are taking germs away from zero and objects have Γ-compact support,

$$\mathcal{A}_{p_{\Gamma}}(X_0)^{\infty} \cong \bigoplus_{j \in J} \Big(\prod_{\Gamma/H_j} \mathcal{A}(\mathrm{pt})^{\infty} \Big),$$

which is a Γ-equivariant equivalence of categories. Let $\mathcal{C}_j = \prod_{\Gamma/H_j} \mathcal{A}(\mathrm{pt})^{\infty}$. Since $\mathbb{K}^{-\infty}(\mathcal{C}_j)$ is Γ-invariant for every $j \in J$,

$$\mathbb{K}^{-\infty}\Big(\bigoplus_{j \in J} \mathcal{C}_j \Big)^{\Gamma} \simeq \Big(\bigvee_{j \in J} \mathbb{K}^{-\infty}(\mathcal{C}_j) \Big)^{\Gamma} \simeq \bigvee_{j \in J} \mathbb{K}^{-\infty}(\mathcal{C}_j)^{\Gamma}$$

and

$$\mathbb{K}^{-\infty}\Big(\bigoplus_{j \in J} \mathcal{C}_j \Big)^{h_{\mathcal{F}}\Gamma} \simeq \Big(\bigvee_{j \in J} \mathbb{K}^{-\infty}(\mathcal{C}_j) \Big)^{h_{\mathcal{F}}\Gamma} \simeq \bigvee_{j \in J} \mathbb{K}^{-\infty}(\mathcal{C}_j)^{h_{\mathcal{F}}\Gamma},$$

where the last weak equivalence makes use of Lemma 7.1 (i). By Lemma 7.2,

$$\mathbb{K}^{-\infty}(\mathcal{C}_j)^{\Gamma} \to \mathbb{K}^{-\infty}(\mathcal{C}_j)^{h_{\mathcal{F}}\Gamma}$$

is a weak homotopy equivalence. Therefore,

$$\bigvee_{j \in J} \mathbb{K}^{-\infty}(\mathcal{C}_j)^{\Gamma} \simeq \bigvee_{j \in J} \mathbb{K}^{-\infty}(\mathcal{C}_j)^{h_{\mathcal{F}}\Gamma},$$

which completes the base case of the induction.

Now assume that the proposition holds for the $(n-1)$-skeleton X_{n-1}. Let $A \to B \to C$ denote the sequence

$$\mathbb{K}^{-\infty}\big(\mathcal{A}_{p_\Gamma}(X_{n-1})^\infty\big) \to \mathbb{K}^{-\infty}\big(\mathcal{A}_{p_\Gamma}(X_n)^\infty\big) \to \mathbb{K}^{-\infty}\big(\mathcal{A}_{p_\Gamma}(X_n, X_{n-1})\big).$$

Consider the following commutative diagram:

$$
\begin{array}{ccccc}
A^\Gamma & \longrightarrow & B^\Gamma & \longrightarrow & C^\Gamma \\
\downarrow a & & \downarrow b & & \downarrow c \\
A^{h_{\mathcal{F}}\Gamma} & \longrightarrow & B^{h_{\mathcal{F}}\Gamma} & \longrightarrow & C^{h_{\mathcal{F}}\Gamma}.
\end{array}
$$

Notice that each row in the diagram is a fibration of spectra. The second row is a fibration since taking homotopy fixed sets and taking homotopy fibers are both homotopy limits and therefore commute [BK72].

We must show that b is a weak homotopy equivalence. By the induction hypothesis, a is a weak homotopy equivalence. Therefore, by the Five Lemma, it suffices to prove that c is a weak homotopy equivalence.

Since we are taking germs away from X^{n-1} (and thus away from zero) and objects have Γ-compact support,

$$\mathcal{A}_{p_\Gamma}(X_n, X_{n-1}) \cong \bigoplus_{i \in I} \Big(\prod_{\Gamma/H_i} \mathcal{A}(D^n, S^{n-1}) \Big).$$

The proof is now completed by arguing as in the beginning of the induction with $\mathcal{C}_i = \prod_{\Gamma/H_i} \mathcal{A}(D^n, S^{n-1})$. $\qquad\square$

Theorem 7.5 (The Descent Principle). *Let Γ be a discrete group, $\mathcal{F}$ the family of finite subgroups of Γ, X a finite dimensional Γ-CW complex, and p_Γ the quotient map $X \to X/\Gamma$. Assume that X admits a Γ-invariant metric such that $K_n\big(\mathcal{A}_{p_\Gamma}^G(X)\big) = 0$ for every integer n, and every $G \in \mathcal{F}$.*

Then the map $H_*^\Gamma(X; \mathbf{K}_\mathcal{A}) \to K_*(\mathcal{A}[\Gamma])$ is a split injection.

Proof. Consider the following commutative diagram of fibration sequences:

$$
\begin{array}{ccccc}
\mathbb{K}^{-\infty}\big(\mathcal{A}_{p_\Gamma}(X)_0\big)^\Gamma & \longrightarrow & \mathbb{K}^{-\infty}\big(\mathcal{A}_{p_\Gamma}(X)\big)^\Gamma & \longrightarrow & \mathbb{K}^{-\infty}\big(\mathcal{A}_{p_\Gamma}(X)^\infty\big)^\Gamma \\
\downarrow a & & \downarrow b & & \downarrow c \\
\mathbb{K}^{-\infty}\big(\mathcal{A}_{p_\Gamma}(X)_0\big)^{h_{\mathcal{F}}\Gamma} & \longrightarrow & \mathbb{K}^{-\infty}\big(\mathcal{A}_{p_\Gamma}(X)\big)^{h_{\mathcal{F}}\Gamma} & \longrightarrow & \mathbb{K}^{-\infty}\big(\mathcal{A}_{p_\Gamma}(X)^\infty\big)^{h_{\mathcal{F}}\Gamma}.
\end{array}
$$

By Proposition 7.4, c is a weak homotopy equivalence. Since each row in the diagram is a fibration, it suffices to show that $\mathbb{K}^{-\infty}\big(\mathcal{A}_{p_\Gamma}(X)\big)^{h_{\mathcal{F}}\Gamma}$ is weakly contractible. But this follows from Lemma 7.1 (i) and the assumption that $K_n\big(\mathcal{A}_{p_\Gamma}^G(X)\big) = 0$ for every integer n. $\qquad\square$

8. Proof of the Main Theorem

Theorem 8.1. *Let Γ be a discrete group and let $\mathcal{A}$ be a small additive category. Assume that there is a finite dimensional Γ-CW model for the universal space for proper Γ-actions, $\underline{E}\Gamma$, and assume that there is a Γ-invariant metric on $\underline{E}\Gamma$ such that $\underline{E}\Gamma$ is a complete proper path metric space that is uniformly $\mathcal{F}in$-contractible and has finite asymptotic dimension. Then the assembly map, $H_*^\Gamma(\underline{E}\Gamma; \mathbb{K}_\mathcal{A}^{-\infty}) \to K_*(\mathcal{A}[\Gamma])$, in algebraic K-theory, is a split injection.*

Some preparations must be made before we can give the proof of Theorem 8.1. Let G be a finite subgroup of Γ. Let

$$G = H_0, \, H_1, \, \ldots, \, H_m = \{e\}$$

contain exactly one subgroup from each conjugacy class of subgroups of G and let the H_i be ordered by cardinality. That is, $|H_i| \geq |H_{i+1}|$.

For each k, $0 \leq k \leq m$, define $\mathcal{S}_k = \big\{ H_i^g \,\big|\, 0 \leq i \leq k, \, g \in G \big\}$ and $Z_k = \underline{E}\Gamma^{\mathcal{S}_k}$. Clearly, $\mathcal{S}_k$ is invariant under conjugation by G. Therefore, Z_k is G-invariant for every k, $0 \leq k \leq m$.

Notation 8.2. For each k, let $p \colon Z_k \to \underline{E}\Gamma/\Gamma$ be the restriction of the quotient map $p_\Gamma \colon \underline{E}\Gamma \to \underline{E}\Gamma/\Gamma$, and let $p_G \colon Z_k/G \to \underline{E}\Gamma/\Gamma$ denote the restriction of the canonical projection $\underline{E}\Gamma/G \to \underline{E}\Gamma/\Gamma$.

Lemma 8.3. *For every k, $0 \leq k \leq m$, and every subgroup H of G, the K-theory of $\mathcal{A}[H]_{p_G}(Z_k/G)$ vanishes.*

Proof. By Lemmas 1.4 and 2.2, Z_k/G is uniformly contractible and has finite asymptotic dimension. Let K be a subset of Z_k/G whose image, $\bar{K}$, in $\underline{E}\Gamma/G$ is compact. If $\alpha > 0$, the image of K^α is $\bar{K}^\alpha$, which is compact since $\underline{E}\Gamma/G$ is a proper metric space. Therefore, we can apply Proposition 6.2 to conclude that the K-theory of $\mathcal{A}[H]_{p_G}(Z_k/G)$ vanishes. $\square$

For the following fact, compare [Ros04, Lemma 7.4].

Lemma 8.4. *For each k, $1 \leq k \leq m$,*

$$\mathcal{A}_p^G(Z_k, Z_{k-1}) \cong \mathcal{A}[H_k]_{p_G}(Z_k/G, Z_{k-1}/G).$$

Proof. Since we are taking germs away from Z_{k-1} and Z_{k-1}/G, every morphism has a representative that is zero on $Z_{k-1} \times [0,1)$ and on $Z_{k-1}/G \times [0,1)$, respectively. Therefore, it is irrelevant what the objects over $Z_{k-1} \times [0,1)$ and $Z_{k-1}/G \times [0,1)$ are. By construction, the stabilizer subgroup of any point not in $Z_{k-1} \times [0,1)$ is a conjugate of H_k. Since $\mathcal{A}_p^G(Z_k, Z_{k-1})$ is a fixed category, the parts of an object over points in the same orbit must be isomorphic modules. Thus, the object M in $\mathcal{A}_p^G(Z_k, Z_{k-1})$ is sent to M', where $M'_{(y,t)}$ (with $(y,t) \notin Z_{k-1}/G \times [0,1)$) is the $\mathcal{A}[H_k]$-module sitting over the point in $\big\{(x,t) \,\big|\, x \in p^{-1}(y)\big\}$ whose stabilizer subgroup is H_k. The inverse of this is to take the $\mathcal{A}[H_k]$-module over (y,t) and use the G-action to spread it around the orbit $\big\{(x,t) \,\big|\, x \in p^{-1}(y)\big\}$. This explains how the objects in $\mathcal{A}_p^G(Z_k, Z_{k-1})$ and $\mathcal{A}[H_k]_{p_G}(Z_k/G, Z_{k-1}/G)$ are identified. To verify (5.10), notice that if K_0 is a subset of Z_k/Γ, then the image of $p_\Gamma^{-1}(K_0)$ under the quotient map $Z_k \to Z_k/G$ is $p^{-1}(K_0)$. Since we are taking germs, the components of a morphism need to become small. Therefore, non-zero components of a morphism have the same isotropy, namely a conjugate of H_k. Furthermore, the equivariance of morphisms in $\mathcal{A}_p^G(Z_k, Z_{k-1})$ implies that there is only one choice when lifting a morphism from $\mathcal{A}[H_k]_{p_G}(Z_k/G, Z_{k-1}/G)$. $\square$

Proof of Theorem 8.1. By the Descent Principle, it suffices to show that the spectrum $\mathbb{K}^{-\infty}\big(\mathcal{A}_{p_\Gamma}^G(\underline{E}\Gamma)\big)$ is weakly contractible for every finite subgroup G of Γ.

Let G be a finite subgroup of Γ and proceed by induction on the filtration

$$\underline{E}\Gamma^G = Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_{m-1} \subseteq Z_m = \underline{E}\Gamma$$

defined above. Since G acts trivially on $\underline{E}\Gamma^G$, $\mathcal{A}_p^G(\underline{E}\Gamma^G)$ is equivalent to $\mathcal{A}[G]_p(\underline{E}\Gamma^G)$. By Lemma 8.3, $\mathbb{K}^{-\infty}\big(\mathcal{A}[G]_p(\underline{E}\Gamma^G)\big)$ is weakly contractible. This completes the base case of the induction.

Assume now that $\mathbb{K}^{-\infty}\big(\mathcal{A}_p^G(Z_{k-1})\big)$ is weakly contractible. We must show that $\mathbb{K}^{-\infty}\big(\mathcal{A}_p^G(Z_k)\big)$ is weakly contractible. Consider the following Karoubi filtration

$$\mathcal{A}_p^G(Z_{k-1}) \to \mathcal{A}_p^G(Z_k) \to \mathcal{A}_p^G(Z_k, Z_{k-1})$$

(see (5.13)), which yields a fibration of spectra after applying $\mathbb{K}^{-\infty}$. By using the induction hypothesis, we need only show that $\mathbb{K}^{-\infty}\big(\mathcal{A}_p^G(Z_k, Z_{k-1})\big)$ is weakly contractible. By Lemma 8.4, $\mathcal{A}_p^G(Z_k, Z_{k-1})$ is equivalent to $\mathcal{A}[H_k]_{p_G}(Z_k/G, Z_{k-1}/G)$, which fits into the Karoubi filtration:

$$\mathcal{A}[H_k]_{p_G}(Z_{k-1}/G) \to \mathcal{A}[H_k]_{p_G}(Z_k/G) \to \mathcal{A}[H_k]_{p_G}(Z_k/G, Z_{k-1}/G).$$

Both $\mathbb{K}^{-\infty}\big(\mathcal{A}[H_k]_{p_G}(Z_{k-1}/G)\big)$ and $\mathbb{K}^{-\infty}\big(\mathcal{A}[H_k]_{p_G}(Z_k/G)\big)$ are weakly contractible by Lemma 8.3. Therefore, $\mathbb{K}^{-\infty}\big(\mathcal{A}[H_k]_{p_G}(Z_k/G, Z_{k-1}/G)\big)$ is also weakly contractible. $\square$

9. L-THEORY

If $\mathcal{A}$ is an additive category with involution and an action of Γ, then there is an assembly map

$$H_*^\Gamma(\underline{E}\Gamma; \mathbf{L}_\mathcal{A}) \to H_*^\Gamma(\mathrm{pt}; \mathbf{L}_\mathcal{A}) = L_*^{\langle -\infty \rangle}(\mathcal{A}[\Gamma])$$

(see [BR05, Section 5]). Here $\mathbf{L}_\mathcal{A}$ is an $\mathrm{Or}(\Gamma)$-spectrum whose value on Γ/H is weakly equivalent to $\mathbb{L}^{-\infty}(\mathcal{A}[H])$, where $\mathbb{L}^{-\infty}(\mathcal{A}[H])$ is the spectrum whose homotopy groups are the *ultimate lower quadratic L-groups* $L_*^{\langle -\infty \rangle}(\mathcal{A}[H])$ (see [Ran92, Chapter 17]). If $\mathcal{A}$ is the category of finitely generated free R-modules for a ring, R, with involution, then $L_*^{\langle -\infty \rangle}(\mathcal{A}[H]) = L_*^{\langle -\infty \rangle}(R[H])$, and the above assembly map is

$$H_*^\Gamma(\underline{E}\Gamma; \mathbf{L}_R) \to H_*^\Gamma(\mathrm{pt}; \mathbf{L}_R) = L_*^{\langle -\infty \rangle}(R[\Gamma]).$$

The following is the L-theory version of Theorem 8.1.

Theorem 9.1. *Let Γ be discrete group and let $\mathcal{A}$ be a small additive category with involution. Assume that there is a finite dimensional Γ-CW-model for the universal space for proper Γ-actions, $\underline{E}\Gamma$, and assume that there is a Γ-invariant metric on $\underline{E}\Gamma$ such that $\underline{E}\Gamma$ is uniformly $\mathcal{F}in$-contractible, is a complete proper path metric space and has finite asymptotic dimension. Assume that for each finite subgroup G there is an $i_0 \in \mathbb{N}$ such that for $i \geq i_0$, $K_{-i}(\mathcal{A}[G]) = 0$, where the involution is forgotten and $\mathcal{A}$ is considered only as an additive category. Then the assembly map, $H_*^\Gamma(\underline{E}\Gamma; \mathbf{L}_\mathcal{A}) \to L_*(\mathcal{A}[\Gamma])$, in L-theory, is a split injection.*

Proof. Everything we did for K-theory also works for L-theory with the exception of Lemma 7.2. As pointed out in Remark 7.3, this lemma will carry over to L-theory if the additional assumption about the vanishing of $K_{-i}(\mathcal{A}[G])$ for large i is made. The rest of the argument proceeds with out further changes. For the required properties of L-theory, see [CP95, Section 4]. $\square$

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